On the non-Degeneracy of Unsatisfiability Proof Graphs produced by SAT Solvers

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Preliminaries

What is SAT ?

Definition

Let $\Phi(a, b, ..)$ be a boolean formula.

Is there an **interpretation** of $(a, b, ..)$ that satisfies Φ ?

In theory

Cook-Levin's Theorem: SAT is **NP-Complete**.

$\Rightarrow$ First problem proved **NP-Complete**

$\Rightarrow$ $P = NP$?

Notations

Literals

A literal $(a, b, \ldots)$ is either a boolean variable x or the negation of a boolean variable $\neg x$

Clauses

A clause C is a disjunction of literals *i.e*:
$C = a \vee b \vee .. \vee z$

Formula

A formula Φ is a conjunction of clauses *i.e*:
$\Phi = C_1 \wedge C_2 \wedge ... \wedge C_m$

Example

$\Phi = (a \vee \neg b) \wedge b \wedge (\neg a \vee \neg b)$

Resolution rule [Robinson '65]

Let C_1 and C_2 two clauses such that:

$$C_1 = a \lor b \lor c \lor \mathrm{d}$$

$$C_2 = \neg d \lor e \lor f$$

We apply the resolution rule on d:

$$\overbrace{(a \lor b \lor c \lor d)}^{C_1} \land \overbrace{(\neg d \lor e \lor f)}^{C_2}$$
$$\vdash a \lor b \lor c \lor e \lor f$$

Resolution rule [Robinson '65]

More formally,

Let C_1 and C_2 be two clauses, the resolution rule gives us:

$$(C_1 \vee x) \wedge (C_2 \vee \neg x) \vdash C_1 \vee C_2$$

We call $C_1 \vee C_2$ the resolvent of $C_1 \vee x$ and $C_2 \vee \neg x$.

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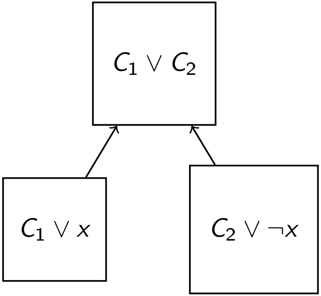


Figure 1: Graphical representation of the resolution

Inconsistency

Correction and completeness of the resolution

→ Φ is an inconsistent formula: Empty clause ($\Box$) can be derivated from the clauses of Φ;

→ An inconsistent proof represents the sequence of resolutions.

In this talk

We will consider only inconsistent proof (Satisfiability proof are trivial).

Modern SAT solver

Φ: Set of initials clauses

Σ: Set of learnts clauses

Algorithm 1 Modern SAT solver

While $\square \notin \Phi \cup \Sigma$ **do**
  $C \leftarrow \mathit{learntClauses}()$
  $\Sigma = \Sigma \cup C$
  If $\mathit{overfull}(\Sigma)$ **Then**
    $\Delta = \mathit{clausesToDelete}(\Sigma)$
    $\Sigma = \Sigma \backslash \Delta$
  End If
End While

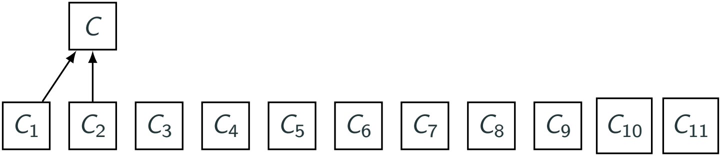
Sequence of resolution with a graph representation

Formula

Let $\Phi = C_1 \wedge C_2 \wedge .. \wedge C_{11}$, s.t. $\forall i \in [1..11]$, C_i any clause.

$$\boxed{C_1} \quad \boxed{C_2} \quad \boxed{C_3} \quad \boxed{C_4} \quad \boxed{C_5} \quad \boxed{C_6} \quad \boxed{C_7} \quad \boxed{C_8} \quad \boxed{C_9} \quad \boxed{C_{10}} \quad \boxed{C_{11}}$$

Sequence of resolution with a graph representation



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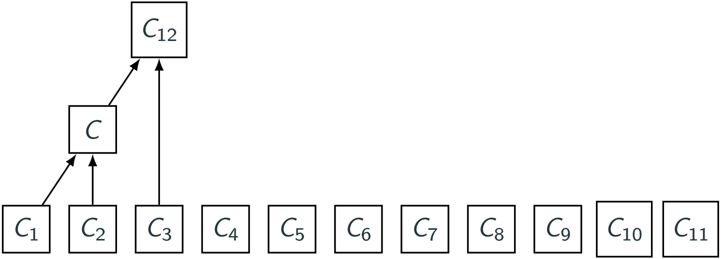

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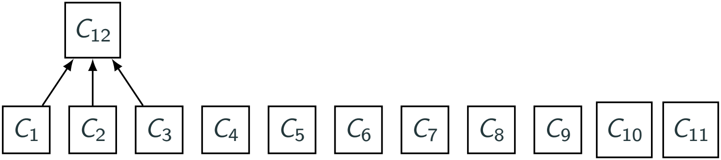

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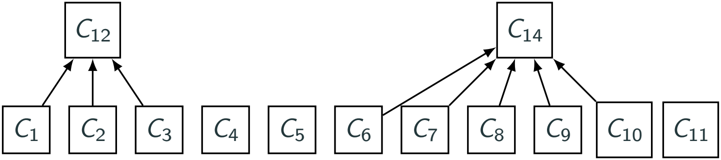

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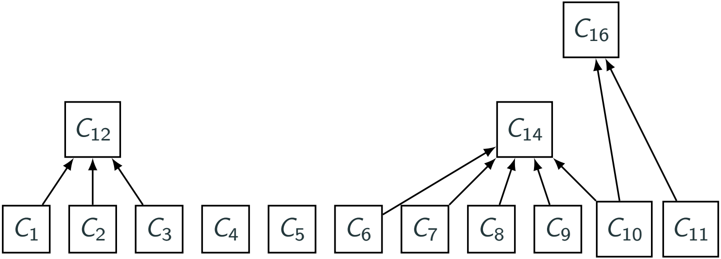

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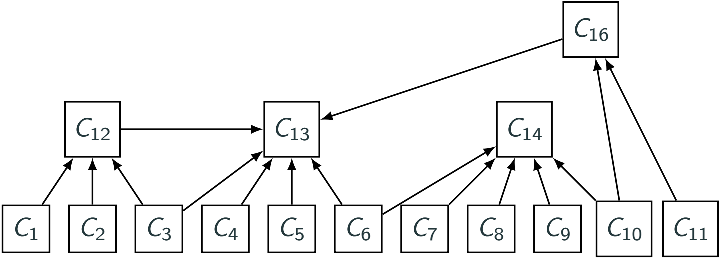

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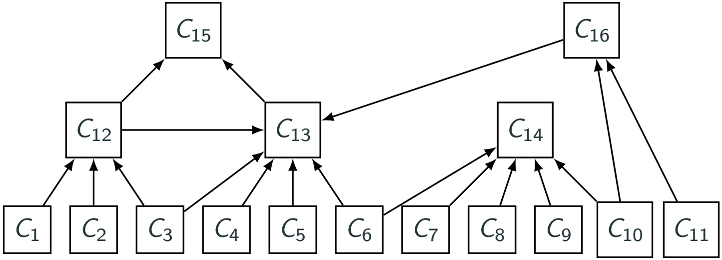

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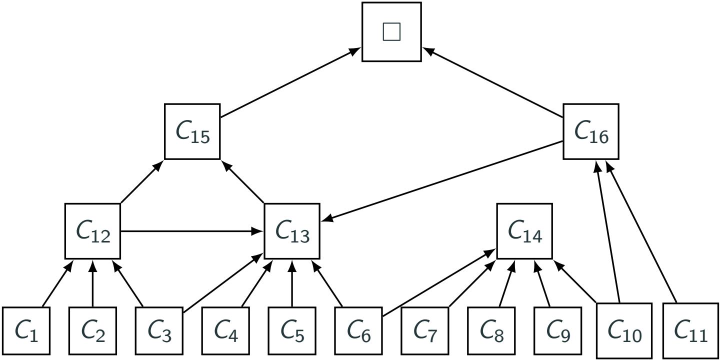
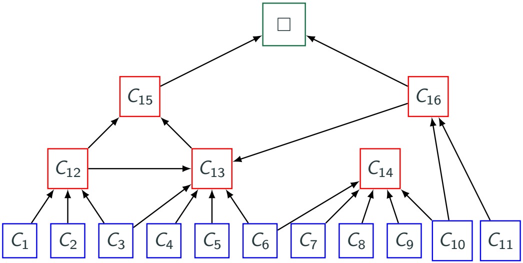


Resolution graph

Definition

The resolution graph is a directed acyclic graph (or **DAG**) such that:

- Leaves are **initials** clauses;
- Internal nodes are **learnts** clauses;
- The root is the **empty** clause.



We call it the proof produced by the SAT solver.

Characterization of K-Cores

Representation of a real proof

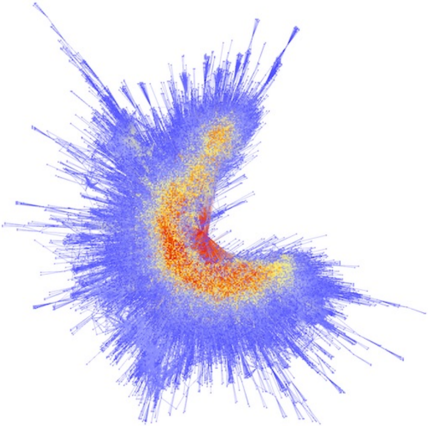


Figure 2: Force-Directed layout of the Dependency Graph for the benchmark `een-pico-prop-05`. The color shows the **degree** of each node.

Information

Formula

\# clauses: 55585

\# variables: 50076

Conflicts

\# conflicts: 59792

CPU time: **6s**

Graph

\# vertices: 51274

\# edges: **960620**

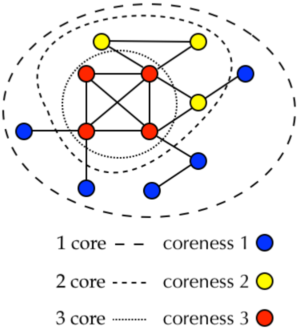
Definitions

K-Core

A *k*-core of a graph G is an undirected subgraph in which all vertices have degree at least *k*.

Coreness

The coreness of a vertex is *k* *iff* he belongs to a *k*-core but not to any *k+1*-core.



Experiments conditions

Selection of `UNSAT` problems

- 60 problems from the 2012-2017 `SAT` competitions;
- Select at least **two** benchmarks per family of problems;
- Need less than one million conflicts to be solved on the original formula.

Conditions

- Cluster of Xeon E7-4870 processors from the *Mesocentre Aquitain de Calcul Intensif*;

Characterization of K-Cores

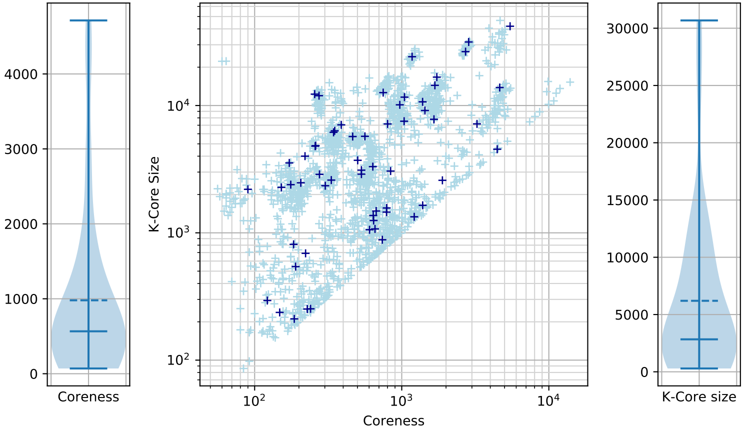


Figure 3: On **left** and **right**, we show the distribution of median values over shuffled instances. In the **middle**, blue darker plots are original problems, lighter shuffled problems.

Characterization of K-Cores

We call **useful** a clause necessary for the proof, and **useless** otherwise.

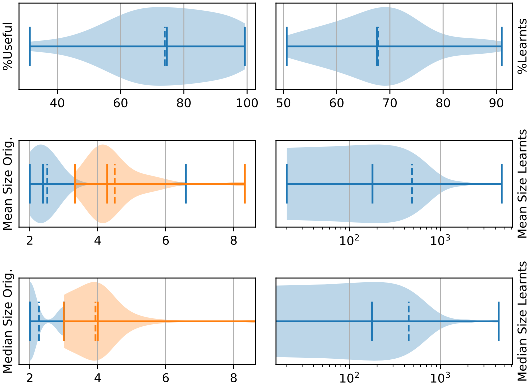
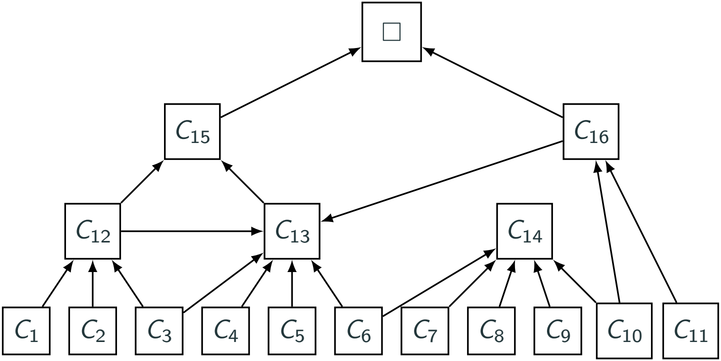


Figure 4: Violin plots summarizing some information about **K-Cores**. All numbers are median values over 50 shuffled and original problems over our set of 60 problems.

Exemple of useless clauses



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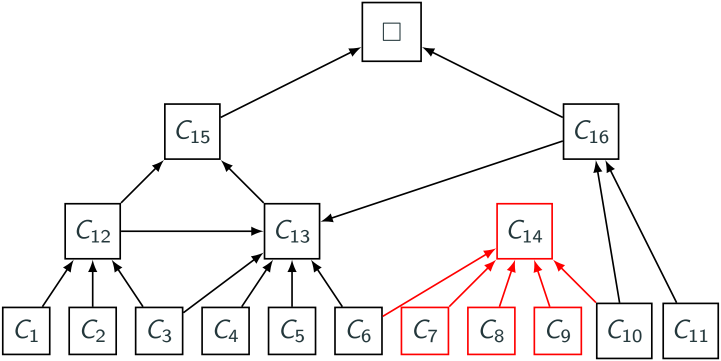

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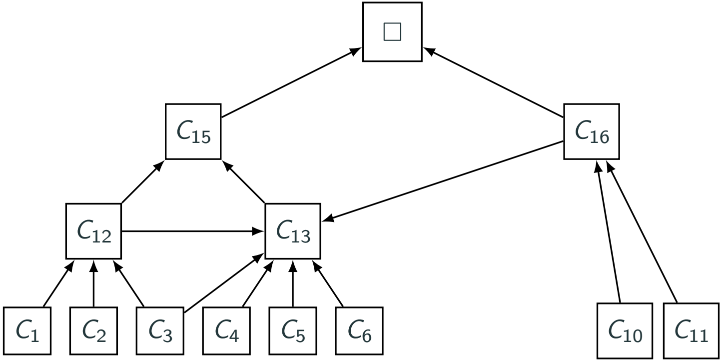

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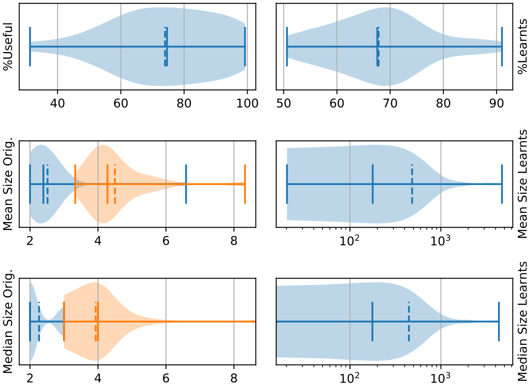

Figure 5: Violin plots summarizing some information about **K-Cores**. All numbers are median values over 50 shuffled and original problems over our set of 60 problems.

Characterization of K-Cores

Summary

- K-Cores can be very large (median larger than **2000**);
- Surprisingly not entirely composed of useful clauses;
- Original K-Core clauses are brief (median of the clauses are binary or ternary clauses);
- However, large disparity in the size of K-Core learnt clauses.

On predictions based on Dependency Graph analysis

On predictions based on Dependency Graph analysis

Objectives

- Identify during the analysis which clauses will be useful;
- Guess which variables will occur in the last learnt clauses, just before deriving the final contradiction.

Conditions

In both, we report the analysis of the **DG** (Dependency Graph) after **20,000 conflicts** and at **half-run**, by simply removing from the **DG** all the nodes and edges added after the limit.

A simple flow algorithm

The idea is to measure the clauses that occur in the **maximal** number of paths to a root node (clause without descendant).

Predicting useful clauses

Predicting useful clauses

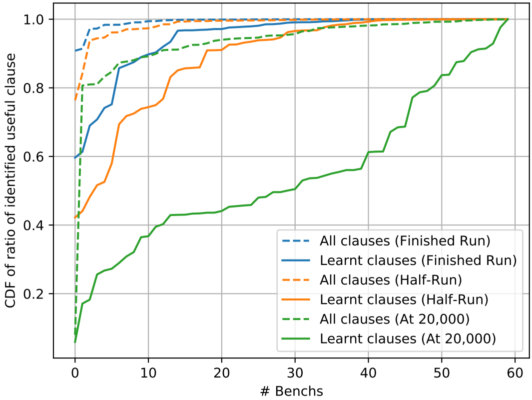

Figure 6: All clauses include 10,000 clauses that can be original or learnt. Learnt clauses restrict the computation to only learnt clauses. A **prediction** is made after 20,000 conflicts, at half-run, and at the end of the run.

Predicting useful clauses

Some results...

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- At half of the computation, we correctly guess at least ***90%*** of useful learnt clauses for **half of the problems**;

... But no improvement yet

Despite the importance of being able to detect a **useful** clause early, it did not allowed us to improve `Glucose` **yet**.

Detecting future learnt clauses

Detecting future learnt clauses

Detection of variables in the last learnt clause

- We want to guess which **variables** will occur in the last learnt clauses;
- We studied the variables occurring in the K-Core at **half of the computation**;
- We only consider **the most frequent variables** in the K-Core.

Detecting future learnt clauses

Results of our prediction for literals occurring over 60 problems

Table - Top-Y variables (*rows*) w.r.t the last X learnt clauses (*columns*)

	20	**50**	**100**	**1000**
5	27	37	45	53
10	42	47	50	54
20	49	51	51	55

Detecting future learnt clauses

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Conclusion

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- We have highlighted the existence of a very dense subgraph in the proofs, the K-Core;
- We are capable of identifying a set of useful learnt clauses at **half** of the run;
- We can also identify a very small set of variables that will occur in the very last learnt clauses;

Further work

- We will try to improve `SAT` solvers thanks to the detection of useful clauses early in computation;
- Take into account the existence of K-Core even for the parallelization;

Thank you

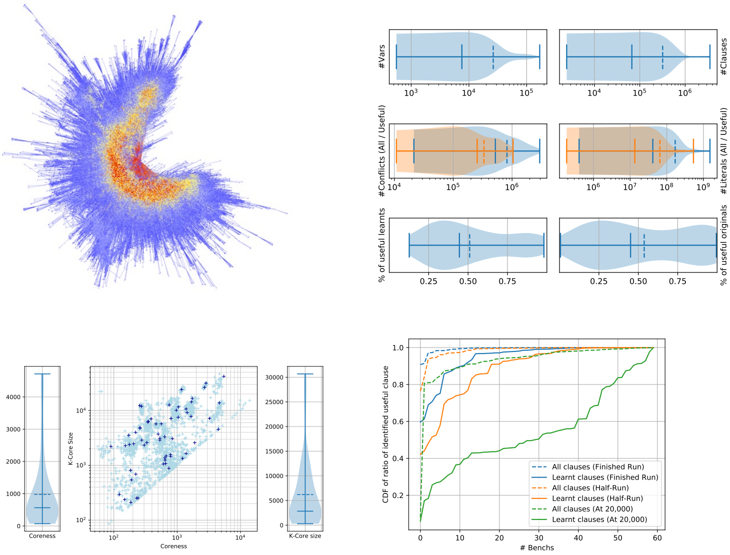

Figure 7: Overview of our results

Analysis of Parallel Proofs

Analysis of Parallel Proofs

- How does the proof evolve *w.r.t* the number of threads ?
- We simulated a Round Robin parallel `SAT` solver, based on `Glucose`

Modern parallel SAT solver

Φ: Set of initials clauses

Σ: Set of learnts clauses

Algorithm 2 Modern parallel SAT solver

While $\square \notin \Phi \cup \Sigma$ **do**

  $C = \mathit{learntClauses}()$

  $\Sigma = \Sigma \cup C$

  $\mathit{Export}(C)$

  $\Sigma = \Sigma \cup \mathit{importClause}()$

  If $\mathit{overfull}(\Sigma)$ **Then**

    $\Delta = \mathit{clausestoDelete}(\Sigma)$

    $\Sigma = \Sigma \backslash \Delta$

  End If

End While

Analysis of Parallel Proofs

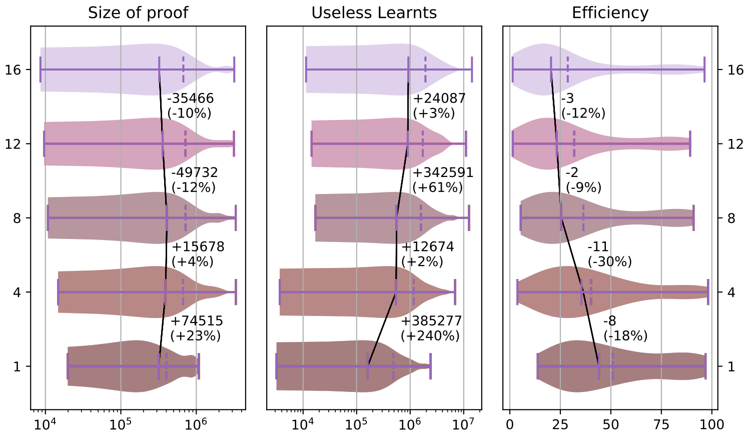

Figure 8: The evolution of metrics on the proofs according to the number of solvers

Analysis of Parallel Proofs

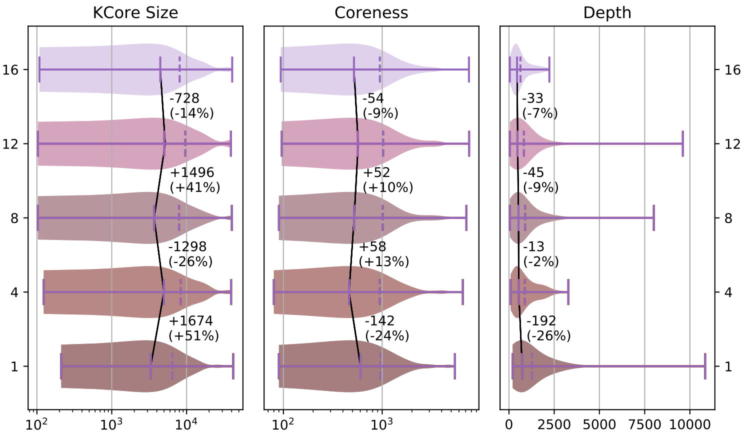

Figure 9: The evolution of metrics on the K-Cores according to the number of solvers

Analysis of Parallel Proofs

Summary

- Parallelization does not seem to have a big impact on **K-Core**;
- It does not seem to have a big impact of parallelization on the K-Core;